

A theory of tensor products for module categories for a vertex operator algebra, III*

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Abstract

This is the third part in a series of papers developing a tensor product theory for modules for a vertex operator algebra. The goal of this theory is to construct a “vertex tensor category” structure on the category of modules for a suitable vertex operator algebra. The notion of vertex tensor category is essentially a “complex analogue” of the notion of symmetric tensor category, and in fact a vertex tensor category produces a braided tensor category in a natural way. In this paper, we focus on a particular element $P(z)$ of a certain moduli space of three-punctured Riemann spheres; in general, every element of this moduli space will give rise to a notion of tensor product, and one must consider all these notions in order to construct a vertex tensor category. Here we present the fundamental properties of the $P(z)$ -tensor product of two modules for a vertex operator algebra. We give two constructions of a $P(z)$ -tensor product, using the results, established in Parts I and II of this series, for a certain other element of the moduli space. The definitions and results in Parts I and II are recalled.

One of the most important operations in the representation theory of Lie algebras is the tensor product operation for modules for a Lie algebra.

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Together with this operation and the coefficient field as an identity object, the category of modules for a Lie algebra is a symmetric tensor category. But for any category of modules of a fixed nonzero level for an affine Lie algebra, the usual tensor product operation does not give a tensor category structure. Instead, there are certain categories of modules of a fixed level for an affine Lie algebra equipped with conformal-field-theoretic tensor product operations, and modules playing the role of identity objects, giving braided tensor categories. From the viewpoint of conformal field theory, the most relevant cases involve positive integral levels, including the case of the category whose objects are finite direct sums of modules isomorphic to standard (integrable highest weight) modules of a fixed positive integral level for an affine Lie algebra. This was explained on a physical level of rigor by Moore and Seiberg [MS], under the very subtle and nontrivial assumption that the “chiral vertex operators” have suitable “operator product expansion” properties, or that an equivalent geometric axiom in conformal field theory holds. In [KL1]–[KL4], Kazhdan and Lusztig constructed the braided tensor category structure for another category of modules, of a fixed but sufficiently negative level, for an affine Lie algebra.

Vertex operator algebras ([B], [FLM], [FHL]) are analogous to both Lie algebras and commutative associative algebras. They are essentially equivalent to “chiral algebras” in conformal field theory (see for example [MS]). Motivated partly by the analogy between vertex operator algebras and Lie algebras and partly by the announcement [KL1], we initiated a theory of tensor products for modules for a vertex operator algebra in [HL1]. This theory was developed in detail beginning in [HL2]–[HL3], and an overview of the theory was given in [HL4], where it was announced that for a vertex operator algebra satisfying suitable conditions, its module category has a natural structure of “vertex tensor category.” It was also announced there that the underlying category of a vertex tensor category has a natural structure of braided tensor category. In particular, the category of modules for the vertex operator algebra associated to a minimal model and the category whose objects are finite direct sums of modules isomorphic to standard modules of a fixed positive integral level for an affine Lie algebra have natural braided tensor category structure. In this theory, instead of being an assumption as in [MS], the operator product expansion (or associativity) for chiral vertex operators (or intertwining operators) is a consequence.

The present paper (Part III) is a continuation of [HL2] (Part I) and

[HL3] (Part II), to which—especially [HL2]—the reader is referred for the necessary background, including references. The reader is referred to [HL4] for the motivation and description of the main results of the tensor product theory developed by the present series of papers. To make the present paper as self-contained as reasonably possible, we shall provide in this introduction a systematic summary of the necessary results of Parts I and II.

In Part I, the notions of $P(z)$ - and $Q(z)$ -tensor product ($z \in \mathbb{C}^\times$) of modules for a vertex operator algebra were introduced, and under suitable conditions, two constructions of a $Q(z)$ -tensor product were given in Part I based on certain results proved in Part II. (The symbols $P(z)$ and $Q(z)$ designate two elements of a certain moduli space of spheres with punctures and local coordinates.) In the present paper, the notion of $P(z)$ -tensor product is discussed in parallel with the discussion in Section 4 of Part I for that of $Q(z)$ -tensor product, and two constructions of a $P(z)$ -tensor product are given, using the results for the $Q(z^{-1})$ -tensor product.

We now describe some of our basic notation and elementary tools. We work over \mathbb{C} . In this paper, as in [HL2] and [HL3], the symbols x, x_0, x_1, \dots and t are independent commuting formal variables, and all expressions involving these variables are to be understood as formal Laurent series. We use the “formal δ -function”

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n.$$

It has the following simple and fundamental property: For any $f(x) \in \mathbb{C}[x, x^{-1}]$,

$$f(x)\delta(x) = f(1)\delta(x).$$

This property has many important variants. For example, for any

$$X(x_1, x_2) \in (\text{End } W)[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$$

(where W is a vector space) such that

$$\lim_{x_1 \rightarrow x_2} X(x_1, x_2) = X(x_1, x_2) \Big|_{x_1=x_2}$$

exists, we have

$$X(x_1, x_2)\delta\left(\frac{x_1}{x_2}\right) = X(x_2, x_2)\delta\left(\frac{x_1}{x_2}\right).$$

The existence of this “algebraic limit” means that for an arbitrary vector $w \in W$, the coefficient of each power of x_2 in the formal expansion $X(x_1, x_2)w \Big|_{x_1=x_2}$ is a finite sum. We use the convention that negative powers of a binomial are to be expanded in nonnegative powers of the second summand. For example,

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) = \sum_{n \in \mathbb{Z}} \frac{(x_1 - x_2)^n}{x_0^{n+1}} = \sum_{m \in \mathbb{N}, n \in \mathbb{Z}} (-1)^m \binom{n}{m} x_0^{-n-1} x_1^{n-m} x_2^m.$$

We have the following identities:

$$\begin{aligned} x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right), \\ x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right). \end{aligned}$$

We shall use these properties and identities later on without explicit comment. Here and below, it is important to note that the relevant sums and products, etc., of formal series, are well defined. See [FLM], [FHL], [HL2] and [HL3] for further discussion and many examples of their use, including their role in formulating and using the Jacobi identity for vertex operator algebras, modules and intertwining operators.

As in [HL2] and [HL3], the symbol z will always denote a nonzero complex number. We shall always choose $\log z$ such that

$$\log z = \log |z| + i \arg z \quad \text{with} \quad 0 \leq \arg z < 2\pi.$$

Arbitrary values of the log function at z will be denoted

$$l_p(z) = \log z + 2p\pi i$$

for $p \in \mathbb{Z}$.

We fix a vertex operator algebra $V (= (V, Y, \mathbf{1}, \omega))$; recall that Y is the vertex operator map; for $v \in V$, $Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$ ($v_n \in \text{End } V$); $\mathbf{1}$ is the vacuum vector; and ω is the element whose vertex operator $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2}$ gives the Virasoro algebra. (See [FLM] and [FHL] for the basic definitions.) The symbols W, W_1, W_2, \dots will denote (\mathbb{C} -graded) V -modules. The symbol Y will denote the vertex operator map for W (as well

as for V), and the symbols Y_1, Y_2, \dots will denote the vertex operator maps for the V -modules W_1, W_2, \dots , respectively. When necessary, we shall use notation such as (W, Y) to designate a V -module. The symbols $\mathcal{Y}, \mathcal{Y}_1, \dots$ will denote intertwining operators (as defined in [FHL]). The vector space of intertwining operators of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ will be denoted $\mathcal{V}_{W_1 W_2}^{W_3}$. The dimension of $\mathcal{V}_{W_1 W_2}^{W_3}$ is the corresponding *fusion rule*. As in [HL2], for any $v \in V$,

$$Y_t(v, x) = v \otimes t^{-1} \delta\left(\frac{x}{t}\right) \in V \otimes \mathbb{C}[[t, t^{-1}, x, x^{-1}]].$$

We also need the notion of generalized module for V . A generalized module for V is a pair (W, Y) satisfying all the axioms for a V -module except for the two axioms on the grading of W (see Definition 2.11 in [HL2]).

Now we present the main results on $Q(z)$ -tensor products, together with the necessary definitions and explanations, from Parts I and II.

Fix a nonzero complex number z and let (W_1, Y_1) and (W_2, Y_2) be V -modules. We recall the definition of $Q(z)$ -tensor product of W_1 and W_2 . A $Q(z)$ -intertwining map of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ is a linear map $F : W_1 \otimes W_2 \rightarrow \overline{W}_3$ (\overline{W} being the formal algebraic completion of a given module W) such that

$$\begin{aligned} z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) Y_3^*(v, x_0) F(w_{(1)} \otimes w_{(2)}) &= \\ &= x_0^{-1} \delta\left(\frac{x_1 - z}{x_0}\right) F(Y_1^*(v, x_1) w_{(1)} \otimes w_{(2)}) \\ &\quad - x_0^{-1} \delta\left(\frac{z - x_1}{-x_0}\right) F(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \end{aligned}$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, where for a vertex operator Y ,

$$Y^*(v, x) = Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}).$$

We denote the vector space of $Q(z)$ -intertwining maps of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ by $\mathcal{M}[Q(z)]_{W_1 W_2}^{W_3}$.

We define a $Q(z)$ -product of W_1 and W_2 to be a V -module (W_3, Y_3) together with a $Q(z)$ -intertwining map F of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ and we denote it by $(W_3, Y_3; F)$ (or (W_3, F)). Let $(W_3, Y_3; F)$ and $(W_4, Y_4; G)$ be two $Q(z)$ -products of W_1 and W_2 . A *morphism* from $(W_3, Y_3; F)$ to $(W_4, Y_4; G)$ is a module map η from W_3 to W_4 such that

$$G = \overline{\eta} \circ F$$

where $\bar{\eta}$ is the natural map from \bar{W}_3 to \bar{W}_4 uniquely extending η . A $Q(z)$ -*tensor product* of W_1 and W_2 is a $Q(z)$ -product $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ such that for any $Q(z)$ -product $(W_3, Y_3; F)$, there is a unique morphism from $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ to $(W_3, Y_3; F)$. The V -module $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)})$ is called a $Q(z)$ -*tensor product module* of W_1 and W_2 . If it exists, it is unique up to canonical isomorphism.

We now describe the close connection between intertwining operators of type $\begin{pmatrix} W'_1 \\ W'_3 W_2 \end{pmatrix}$ and $Q(z)$ -intertwining maps of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$, where W' denotes the contragredient of a module W , as defined in [FHL] (or, more generally, the graded dual space of a graded vector space). Fix an integer p . Let \mathcal{Y} be an intertwining operator of type $\begin{pmatrix} W'_1 \\ W'_3 W_2 \end{pmatrix}$. We have a linear map $F_{\mathcal{Y}, p}^{Q(z)} : W_1 \otimes W_2 \rightarrow \bar{W}_3$ determined by the condition

$$\langle w'_{(3)}, F_{\mathcal{Y}, p}^{Q(z)}(w_{(1)} \otimes w_{(2)}) \rangle_{W_3} = \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, e^{l_p(z)} w_{(2)}) \rangle_{W'_1} \quad (\text{I.1})$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W'_3$, where for a module W , $\langle \cdot, \cdot \rangle_W$ denotes the canonical pairing between W' and \bar{W} , and where the notation $\mathcal{Y}(\cdot, e^\zeta)$ for $\zeta \in \mathbb{C}$ is shorthand for $\mathcal{Y}(\cdot, x)|_{x^n = e^{n\zeta}, n \in \mathbb{C}}$, which is well defined; note that $\mathcal{Y}(\cdot, e^\zeta)$ actually depends on ζ and not just e^ζ . Using the Jacobi identity for \mathcal{Y} , we see easily that $F_{\mathcal{Y}, p}^{Q(z)}$ is a $Q(z)$ -intertwining map of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$.

Conversely, given a $Q(z)$ -intertwining map F of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$, as a linear map from $W_1 \otimes W_2$ to \bar{W}_3 it gives us an element of $(W_1 \otimes W'_3 \otimes W_2)^*$ whose value at $w_{(1)} \otimes w'_{(3)} \otimes w_{(2)}$ is

$$\langle w'_{(3)}, F(w_{(1)} \otimes w_{(2)}) \rangle_{W_3}.$$

But since every element of $(W_1 \otimes W'_3 \otimes W_2)^*$ also amounts to a linear map from $W'_3 \otimes W_2$ to W_1^* , we have such a map as well. Let $w'_{(3)} \in W'_3$ and $w_{(2)} \in W_2$ be homogeneous elements. Since $W_1^* = \prod_{n \in \mathbb{C}} (W'_1)_{(n)}$, the image of $w'_{(3)} \otimes w_{(2)}$ under our map can be written as $\sum_{n \in \mathbb{C}} (w'_{(3)})_n w_{(2)} e^{(-n-1)l_p(z)}$ where for any $n \in \mathbb{C}$, $(w'_{(3)})_n w_{(2)} e^{(-n-1)l_p(z)}$ is the projection of the image to the homogeneous subspace of W'_1 of weight equal to

$$\text{wt } w'_{(3)} - n - 1 + \text{wt } w_{(2)}.$$

(Here we are defining elements denoted $(w'_{(3)})_n w_{(2)}$ of W'_1 for $n \in \mathbb{C}$.) We define

$$\mathcal{Y}_{F,p}(w'_{(3)}, x)w_{(2)} = \sum_{n \in \mathbb{C}} (w'_{(3)})_n w_{(2)} x^{-n-1} \in W'_1\{x\}$$

for all homogeneous elements $w'_{(3)} \in W'_3$ and $w_{(2)} \in W_2$. (For a vector space W , we use the notation

$$W\{x\} = \left\{ \sum_{n \in \mathbb{C}} a_n x^n \mid a_n \in W, n \in \mathbb{C} \right\};$$

in particular, we are allowing complex powers of our commuting formal variables.) Using linearity, we extend $\mathcal{Y}_{F,p}$ to a linear map

$$\begin{aligned} W'_3 \otimes W_2 &\rightarrow W'_1\{x\} \\ w'_{(3)} \otimes w_{(2)} &\mapsto \mathcal{Y}_{F,p}(w'_{(3)}, x)w_{(2)}. \end{aligned}$$

The correspondence $F \mapsto \mathcal{Y}_{F,p}$ is linear, and we have $\mathcal{Y}_{F_{\mathcal{Y},p}^{Q(z)},p} = \mathcal{Y}$ for an intertwining operator \mathcal{Y} of type $\left(\begin{smallmatrix} W'_1 \\ W'_3 W_2 \end{smallmatrix} \right)$.

Proposition I.1 *For $p \in \mathbb{Z}$, the correspondence $\mathcal{Y} \mapsto F_{\mathcal{Y},p}^{Q(z)}$ is a linear isomorphism from the space $\mathcal{V}_{W'_3 W_2}^{W'_1}$ of intertwining operators of type $\left(\begin{smallmatrix} W'_1 \\ W'_3 W_2 \end{smallmatrix} \right)$ to the space $\mathcal{M}[Q(z)]_{W_1 W_2}^{W_3}$ of $Q(z)$ -intertwining maps of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix} \right)$. Its inverse is given by $F \mapsto \mathcal{Y}_{F,p}$.*

The following immediate result relates module maps from a tensor product module with intertwining maps and intertwining operators:

Proposition I.2 *Suppose that $W_1 \boxtimes_{Q(z)} W_2$ exists. We have a natural isomorphism*

$$\begin{aligned} \text{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) &\xrightarrow{\sim} \mathcal{M}_{W_1 W_2}^{W_3} \\ \eta &\mapsto \bar{\eta} \circ \boxtimes_{Q(z)} \end{aligned}$$

and for $p \in \mathbb{Z}$, a natural isomorphism

$$\begin{aligned} \text{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) &\xrightarrow{\sim} \mathcal{V}_{W'_3 W_2}^{W'_1} \\ \eta &\mapsto \mathcal{Y}_{\eta,p} \end{aligned}$$

where $\mathcal{Y}_{\eta,p} = \mathcal{Y}_{F,p}$ with $F = \bar{\eta} \circ \boxtimes_{Q(z)}$.

We have:

Proposition I.3 *For any integer r , there is a natural isomorphism*

$$B_r : \mathcal{V}_{W_1 W_2}^{W_3} \rightarrow \mathcal{V}_{W'_3 W_2}^{W'_1}$$

defined by the condition that for any intertwining operator \mathcal{Y} in $\mathcal{V}_{W_1 W_2}^{W_3}$ and $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W'_3$,

$$\begin{aligned} \langle w_{(1)}, B_r(\mathcal{Y})(w'_{(3)}, x)w_{(2)} \rangle_{W'_1} &= \\ &= \langle e^{-x^{-1}L(1)}w'_{(3)}, \mathcal{Y}(e^{xL(1)}w_{(1)}, x^{-1}) \cdot \\ &\quad \cdot e^{-xL(1)}e^{(2r+1)\pi i L(0)}x^{-2L(0)}w_{(2)} \rangle_{W_3}. \end{aligned}$$

The last two results give:

Corollary I.4 *For any V -modules W_1, W_2, W_3 such that $W_1 \boxtimes_{Q(z)} W_2$ exists and any integers p and r , we have a natural isomorphism*

$$\begin{aligned} \text{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) &\xrightarrow{\sim} \mathcal{V}_{W_1 W_2}^{W_3} \\ \eta &\mapsto B_r^{-1}(\mathcal{Y}_{\eta, p}). \end{aligned}$$

It is clear from the definition of $Q(z)$ -tensor product that the $Q(z)$ -tensor product operation distributes over direct sums in the following sense:

Proposition I.5 *For V -modules $U_1, \dots, U_k, W_1, \dots, W_l$, suppose that each $U_i \boxtimes_{Q(z)} W_j$ exists. Then $(\coprod_i U_i) \boxtimes_{Q(z)} (\coprod_j W_j)$ exists and there is a natural isomorphism*

$$\left(\coprod_i U_i \right) \boxtimes_{Q(z)} \left(\coprod_j W_j \right) \xrightarrow{\sim} \coprod_{i,j} U_i \boxtimes_{Q(z)} W_j.$$

Now consider V -modules W_1, W_2 and W_3 and suppose that

$$\dim \mathcal{M}[Q(z)]_{W_1 W_2}^{W_3} < \infty.$$

The natural evaluation map

$$\begin{aligned} W_1 \otimes W_2 \otimes \mathcal{M}[Q(z)]_{W_1 W_2}^{W_3} &\rightarrow \overline{W}_3 \\ w_{(1)} \otimes w_{(2)} \otimes F &\mapsto F(w_{(1)} \otimes w_{(2)}) \end{aligned}$$

gives a natural map

$$\mathcal{F}[Q(z)]_{W_1 W_2}^{W_3} : W_1 \otimes W_2 \rightarrow \text{Hom}(\mathcal{M}[Q(z)]_{W_1 W_2}^{W_3}, \overline{W}_3) = (\mathcal{M}[Q(z)]_{W_1 W_2}^{W_3})^* \otimes \overline{W}_3.$$

Also, $(\mathcal{M}[Q(z)]_{W_1 W_2}^{W_3})^* \otimes W_3$ is a V -module (with finite-dimensional weight spaces) in the obvious way, and the map $\mathcal{F}[Q(z)]_{W_1 W_2}^{W_3}$ is clearly a $Q(z)$ -intertwining map, where we make the identification

$$(\mathcal{M}[Q(z)]_{W_1 W_2}^{W_3})^* \otimes \overline{W}_3 = \overline{(\mathcal{M}[Q(z)]_{W_1 W_2}^{W_3})^* \otimes W_3}.$$

This gives us a natural $Q(z)$ -product.

Now we consider a special but important class of vertex operator algebras satisfying certain finiteness and semisimplicity conditions.

Definition I.6 A vertex operator algebra V is *rational* if it satisfies the following conditions:

1. There are only finitely many irreducible V -modules (up to equivalence).
2. Every V -module is completely reducible (and is in particular a *finite* direct sum of irreducible modules).
3. All the fusion rules for V are finite (for triples of irreducible modules and hence arbitrary modules).

The next result shows that $Q(z)$ -tensor products exist for the category of modules for a rational vertex operator algebra.

Proposition I.7 *Let V be rational and let W_1, W_2 be V -modules. Then $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ exists, and in fact*

$$W_1 \boxtimes_{Q(z)} W_2 = \coprod_{i=1}^k (\mathcal{M}[Q(z)]_{W_1 W_2}^{M_i})^* \otimes M_i,$$

where $\{M_1, \dots, M_k\}$ is a set of representatives of the equivalence classes of irreducible V -modules, and the right-hand side is equipped with the V -module and $Q(z)$ -product structure indicated above. That is,

$$\boxtimes_{Q(z)} = \sum_{i=1}^k \mathcal{F}[Q(z)]_{W_1 W_2}^{M_i}.$$

This construction of a $Q(z)$ -tensor product module is essentially tautological. We now describe two constructions, which are more useful, of a $Q(z)$ -tensor product of two modules for a vertex operator algebra V , in the presence of a certain hypothesis which holds in case V is rational. Fix a nonzero complex number z and V -modules (W_1, Y_1) and (W_2, Y_2) as before. We first define a linear action of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on $(W_1 \otimes W_2)^*$, that is, a linear map

$$\tau_{Q(z)} : V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}] \rightarrow \text{End } (W_1 \otimes W_2)^*,$$

where ι_+ is the operation of expanding a rational function in the formal variable t in the direction of positive powers of t (with at most finitely many negative powers of t), by

$$\begin{aligned} & \left(\tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) \\ &= x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) \lambda(Y_1^*(v, x_1) w_{(1)} \otimes w_{(2)}) \\ & \quad - x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}). \end{aligned}$$

(Recall that Y_t and Y^* have been defined above, and note that the coefficients of the monomials in x_0 and x_1 in

$$z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0),$$

for all $v \in V$, span the space $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$.) Write

$$Y'_{Q(z)}(v, x) = \tau_{Q(z)}(Y_t(v, x)).$$

Proposition I.8 *The action $Y'_{Q(z)}$ satisfies the commutator formula for vertex operators, that is, on $(W_1 \otimes W_2)^*$,*

$$\begin{aligned} & [Y'_{Q(z)}(v_1, x_1), Y'_{Q(z)}(v_2, x_2)] = \\ &= \text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y'_{Q(z)}(Y(v_1, x_0) v_2, x_2) \end{aligned}$$

for $v_1, v_2 \in V$.

(The symbol Res_x designates the coefficient of x^{-1} in a formal series.)

Let W_3 be another V -module. Note that $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ acts on W'_3 in the obvious way. The following result provides some important motivation for the definition of our action on $(W_1 \otimes W_2)^*$:

Proposition I.9 *Under the natural isomorphism*

$$\text{Hom}(W'_3, (W_1 \otimes W_2)^*) \xrightarrow{\sim} \text{Hom}(W_1 \otimes W_2, \overline{W}_3), \quad (\text{I.2})$$

the maps in $\text{Hom}(W'_3, (W_1 \otimes W_2)^)$ intertwining the two actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on W'_3 and $(W_1 \otimes W_2)^*$ correspond exactly to the $Q(z)$ -intertwining maps of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$.*

Remark I.10 Combining the last result with Proposition I.1, we see that the maps in $\text{Hom}(W'_3, (W_1 \otimes W_2)^*)$ intertwining the two actions on W'_3 and $(W_1 \otimes W_2)^*$ also correspond exactly to the intertwining operators of type $\begin{pmatrix} W'_1 \\ W'_3 W_2 \end{pmatrix}$. In particular, given any integer p , the map $(F_{\mathcal{Y},p}^{Q(z)})' : W'_3 \rightarrow (W_1 \otimes W_2)^*$ defined by

$$(F_{\mathcal{Y},p}^{Q(z)})'(w'_{(3)})(w_{(1)} \otimes w_{(2)}) = \langle w_{(1)}, \mathcal{Y}(w'_{(3)}, e^{l_p(z)})w_{(2)} \rangle_{W'_1}$$

(recall (I.1)) intertwines the action of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on W'_3 and the action $\tau_{Q(z)}$ of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$ on $(W_1 \otimes W_2)^*$.

Consider the following nontrivial and subtle *compatibility condition* on $\lambda \in (W_1 \otimes W_2)^*$: The formal Laurent series $Y'_{Q(z)}(v, x_0)\lambda$ involves only finitely many negative powers of x_0 and

$$\begin{aligned} \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) \lambda = \\ = z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y'_{Q(z)}(v, x_0) \lambda \quad \text{for all } v \in V. \end{aligned} \quad (\text{I.3})$$

(Note that the two sides are not *a priori* equal for general $\lambda \in (W_1 \otimes W_2)^*$.)

Let W be a subspace of $(W_1 \otimes W_2)^*$. We say that W is *compatible* for $\tau_{Q(z)}$ if every element of W satisfies the compatibility condition. Also, we say that W is (\mathbb{C}) -graded if it is \mathbb{C} -graded by its weight subspaces, and that W is a *V-module* (respectively, *generalized module*) if W is graded and is a module

(respectively, generalized module) when equipped with this grading and with the action of $Y'_{Q(z)}(\cdot, x)$. (The notion of “weight” for $(W_1 \otimes W_2)^*$ is defined by means of the eigenvalues of the operator $L'_{Q(z)}(0) = \text{Res}_x Y'_{Q(z)}(\omega, x)$; recall that ω is the element of V giving the Virasoro algebra.) A sum of compatible modules or generalized modules is clearly a generalized module. The weight subspace of a subspace W with weight $n \in \mathbb{C}$ will be denoted $W_{(n)}$.

Define

$$W_1 \boxtimes_{Q(z)} W_2 = \sum_{W \in \mathcal{W}_{Q(z)}} W = \bigcup_{W \in \mathcal{W}_{Q(z)}} W \subset (W_1 \otimes W_2)^*,$$

where $\mathcal{W}_{Q(z)}$ is the set all compatible modules for $\tau_{Q(z)}$ in $(W_1 \otimes W_2)^*$. Then $W_1 \boxtimes_{Q(z)} W_2$ is a compatible generalized module. We have:

Proposition I.11 *The subspace $W_1 \boxtimes_{Q(z)} W_2$ of $(W_1 \otimes W_2)^*$ is a generalized module with the following property: Given any V -module W_3 , there is a natural linear isomorphism determined by (I.2) between the space of all $Q(z)$ -intertwining maps of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ and the space of all maps of generalized modules from W'_3 to $W_1 \boxtimes_{Q(z)} W_2$.*

Proposition I.12 *Let V be a rational vertex operator algebra and W_1, W_2 two V -modules. Then $W_1 \boxtimes_{Q(z)} W_2$ is a module.*

Now we assume that $W_1 \boxtimes_{Q(z)} W_2$ is a module (which occurs if V is rational, by the last proposition). In this case, we define a V -module $W_1 \boxtimes_{Q(z)} W_2$ by

$$W_1 \boxtimes_{Q(z)} W_2 = (W_1 \boxtimes_{Q(z)} W_2)' \quad (\text{I.4})$$

(note that by our choice of notation, $\boxtimes' = \boxtimes$) and we write the corresponding action as $Y_{Q(z)}$. Applying Proposition I.11 to the special module $W_3 = W_1 \boxtimes_{Q(z)} W_2$ and the natural isomorphism from the double contragredient module $W'_3 = (W_1 \boxtimes_{Q(z)} W_2)''$ to $W_1 \boxtimes_{Q(z)} W_2$ (recall [FHL], Theorem 5.3.1), we obtain using (I.2) a canonical $Q(z)$ -intertwining map of type $\begin{pmatrix} W_1 \boxtimes_{Q(z)} W_2 \\ W_1 W_2 \end{pmatrix}$, which we denote

$$\begin{aligned} \boxtimes_{Q(z)} : W_1 \otimes W_2 &\rightarrow \overline{W_1 \boxtimes_{Q(z)} W_2} \\ w_{(1)} \otimes w_{(2)} &\mapsto w_{(1)} \boxtimes_{Q(z)} w_{(2)}. \end{aligned}$$

This is the unique linear map such that

$$\langle \lambda, w_{(1)} \boxtimes_{Q(z)} w_{(2)} \rangle_{W_1 \boxtimes_{Q(z)} W_2} = \lambda(w_{(1)} \otimes w_{(2)})$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $\lambda \in W_1 \boxtimes_{Q(z)} W_2$. Moreover, we have:

Proposition I.13 *The $Q(z)$ -product $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ is a $Q(z)$ -tensor product of W_1 and W_2 .*

More generally, dropping the assumption that $W_1 \boxtimes_{Q(z)} W_2$ is a module, we have:

Proposition I.14 *The $Q(z)$ -tensor product of W_1 and W_2 exists (and is given by (I.4)) if and only if $W_1 \boxtimes_{Q(z)} W_2$ is a module.*

It is not difficult to see that any element of $W_1 \boxtimes_{Q(z)} W_2$ is an element λ of $(W_1 \otimes W_2)^*$ satisfying:

The compatibility condition (recall (I.3)): **(a)** The *lower truncation condition*: For all $v \in V$, the formal Laurent series $Y'_{Q(z)}(v, x)\lambda$ involves only finitely many negative powers of x .

(b) The following formula holds:

$$\begin{aligned} \tau_{Q(z)} \left(z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) \lambda = \\ = z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) Y'_{Q(z)}(v, x_0) \lambda \quad \text{for all } v \in V. \end{aligned}$$

The local grading-restriction condition: **(a)** The *grading condition*: λ is a (finite) sum of weight vectors of $(W_1 \otimes W_2)^*$.

(b) Let W_λ be the smallest subspace of $(W_1 \otimes W_2)^*$ containing λ and stable under the component operators $\tau_{Q(z)}(v \otimes t^n)$ of the operators $Y'_{Q(z)}(v, x)$ for $v \in V$, $n \in \mathbb{Z}$. Then the weight spaces $(W_\lambda)_{(n)}$, $n \in \mathbb{C}$, of the (graded) space W_λ have the properties

$$\begin{aligned} \dim (W_\lambda)_{(n)} &< \infty \quad \text{for } n \in \mathbb{C}, \\ (W_\lambda)_{(n)} &= 0 \quad \text{for } n \text{ whose real part is sufficiently small.} \end{aligned}$$

We have the following basic “second construction” of $W_1 \boxtimes_{Q(z)} W_2$:

Theorem I.15 *The subspace of $(W_1 \otimes W_2)^*$ consisting of the elements satisfying the compatibility condition and the local grading-restriction condition, equipped with $Y'_{Q(z)}$, is a generalized module and is equal to $W_1 \boxtimes_{Q(z)} W_2$.*

The following result follows immediately from Proposition I.14, the theorem above and the definition of $W_1 \boxtimes_{Q(z)} W_2$:

Corollary I.16 *The $Q(z)$ -tensor product of W_1 and W_2 exists if and only if the subspace of $(W_1 \otimes W_2)^*$ consisting of the elements satisfying the compatibility condition and the local grading-restriction condition, equipped with $Y'_{Q(z)}$, is a module. In this case, this module coincides with the module $W_1 \boxtimes_{Q(z)} W_2$, and the contragredient module of this module, equipped with the $Q(z)$ -intertwining map $\boxtimes_{Q(z)}$, is a $Q(z)$ -tensor product of W_1 and W_2 , equal to the structure $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ constructed above.*

From this result and Propositions I.12 and I.13, we have:

Corollary I.17 *Let V be a rational vertex operator algebra and W_1, W_2 two V -modules. Then the $Q(z)$ -tensor product $(W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})$ may be constructed as described in Corollary I.16.*

This finishes our review of the results on $Q(z)$ -tensor products, from Parts I and II.

The numberings of sections, formulas, etc., in Part III continue those of Parts I and II.

Part III contains two sections following this introduction. In Section 12, we recall the notion of $P(z)$ -tensor product introduced in Part I, and we present the basic properties of this tensor product analogous to those of the $Q(z)$ -tensor product, including the existence of $P(z)$ -tensor products in the case of rational vertex operator algebras. We only state the results since all the proofs are exactly analogous to those for $Q(z)$ -tensor products (see Parts I and II). We establish two constructions of a $P(z)$ -tensor product, analogous to those of the $Q(z)$ -tensor product, in Section 13.

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12 Properties of $P(z)$ -tensor products

We first review the notion of $P(z)$ -tensor product introduced in [HL2]. Fix $z \in \mathbb{C}^\times$. Recall that V is a fixed vertex operator algebra and let (W_1, Y_1) , (W_2, Y_2) and (W_3, Y_3) be V -modules. By a $P(z)$ -intertwining map of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ we mean a linear map $F : W_1 \otimes W_2 \rightarrow \overline{W}_3$ satisfying the condition

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - z}{x_0} \right) Y_3(v, x_1) F(w_{(1)} \otimes w_{(2)}) &= \\ &= z^{-1} \delta \left(\frac{x_1 - x_0}{z} \right) F(Y_1(v, x_0) w_{(1)} \otimes w_{(2)}) \\ &\quad + x_0^{-1} \delta \left(\frac{z - x_1}{-x_0} \right) F(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \end{aligned} \quad (12.1)$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. (The expressions in (12.1) are well defined, as explained in Part I.)

We denote the vector space of $P(z)$ -intertwining maps of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ by $\mathcal{M}[P(z)]_{W_1 W_2}^{W_3}$.

A $P(z)$ -product of W_1 and W_2 is a V -module (W_3, Y_3) equipped with a $P(z)$ -intertwining map F of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$. We denote it by $(W_3, Y_3; F)$ (or simply by (W_3, F)). Let $(W_4, Y_4; G)$ be another $P(z)$ -product of W_1 and W_2 . A *morphism* from $(W_3, Y_3; F)$ to $(W_4, Y_4; G)$ is a module map η from W_3 to W_4 such that

$$G = \overline{\eta} \circ F, \quad (12.2)$$

where $\overline{\eta}$ is the natural map from \overline{W}_3 to \overline{W}_4 uniquely extending η .

Definition 12.1 A $P(z)$ -tensor product of W_1 and W_2 is a $P(z)$ -product

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$$

such that for any $P(z)$ -product $(W_3, Y_3; F)$, there is a unique morphism from

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$$

to $(W_3, Y_3; F)$. The V -module $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ is called a $P(z)$ -*tensor product module* of W_1 and W_2 .

It is clear that a $P(z)$ -tensor product of W_1 and W_2 is unique up to unique isomorphism if it exists.

We now describe the precise connection between intertwining operators and $P(z)$ -intertwining maps of the same type. Fix an integer p . Let \mathcal{Y} be an intertwining operator of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix}\right)$. We have a linear map $F_{\mathcal{Y}, p}^{P(z)} : W_1 \otimes W_2 \rightarrow \overline{W}_3$ given by

$$F_{\mathcal{Y}, p}^{P(z)}(w_{(1)} \otimes w_{(2)}) = \mathcal{Y}(w_{(1)}, e^{l_p(z)} w_{(2)}) \quad (12.3)$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. Using the Jacobi identity for \mathcal{Y} , we see easily that $F_{\mathcal{Y}, p}^{P(z)}$ is a $P(z)$ -intertwining map. Conversely, given a $P(z)$ -intertwining map F , homogeneous elements $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ and $n \in \mathbb{C}$, we define $(w_{(1)})_n w_{(2)}$ to be the projection of the image of $w_{(1)} \otimes w_{(2)}$ under F to the homogeneous subspace of W_3 of weight

$$\text{wt } w_{(1)} - n - 1 + \text{wt } w_{(2)},$$

multiplied by $e^{(n+1)l_p(z)}$. Using this, we define

$$\mathcal{Y}_{F, p}(w_{(1)}, x)w_{(2)} = \sum_{n \in \mathbb{C}} (w_{(1)})_n w_{(2)} x^{-n-1}, \quad (12.4)$$

and by linearity, we obtain a linear map

$$\begin{aligned} W_1 \otimes W_2 &\rightarrow W_3\{x\} \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}_{F, p}(w_{(1)}, x)w_{(2)}. \end{aligned}$$

The proof of the following result is analogous to (and slightly shorter than) the proof of the corresponding result for $Q(z)$ -intertwining maps in [HL2]:

Proposition 12.2 For $p \in \mathbb{Z}$, the correspondence $\mathcal{Y} \mapsto F_{\mathcal{Y},p}^{P(z)}$ is a linear isomorphism from the vector space $\mathcal{V}_{W_1 W_2}^{W_3}$ of intertwining operators of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ to the vector space $\mathcal{M}[P(z)]_{W_1 W_2}^{W_3}$ of $P(z)$ -intertwining maps of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$. Its inverse map is given by $F \mapsto \mathcal{Y}_{F,p}$. \square

The following immediate result relates module maps from a $P(z)$ -tensor product module with intertwining maps and intertwining operators:

Proposition 12.3 Suppose that $W_1 \boxtimes_{P(z)} W_2$ exists. We have a natural isomorphism

$$\begin{aligned} \text{Hom}_V(W_1 \boxtimes_{P(z)} W_2, W_3) &\xrightarrow{\sim} \mathcal{M}[P(z)]_{W_1 W_2}^{W_3} \\ \eta &\mapsto \bar{\eta} \circ \boxtimes_{P(z)} \end{aligned} \quad (12.5)$$

and for $p \in \mathbb{Z}$, a natural isomorphism

$$\begin{aligned} \text{Hom}_V(W_1 \boxtimes_{P(z)} W_2, W_3) &\xrightarrow{\sim} \mathcal{V}_{W_1 W_2}^{W_3} \\ \eta &\mapsto \mathcal{Y}_{\eta,p} \end{aligned} \quad (12.6)$$

where $\mathcal{Y}_{\eta,p} = \mathcal{Y}_{F,p}$ with $F = \bar{\eta} \circ \boxtimes_{P(z)}$. \square

It is clear from the Definition 12.1 that $P(z)$ -tensor product that the $P(z)$ -tensor product operation distributes over direct sums in the following sense:

Proposition 12.4 Let $U_1, \dots, U_k, W_1, \dots, W_l$ be V -modules and suppose that each $U_i \boxtimes_{P(z)} W_j$ exists. Then $(\coprod_i U_i) \boxtimes_{P(z)} (\coprod_j W_j)$ exists and there is a natural isomorphism

$$\left(\coprod_i U_i \right) \boxtimes_{P(z)} \left(\coprod_j W_j \right) \xrightarrow{\sim} \coprod_{i,j} U_i \boxtimes_{P(z)} W_j. \quad \square \quad (12.7)$$

Now consider V -modules W_1, W_2 and W_3 and suppose that

$$\dim \mathcal{V}_{W_1 W_2}^{W_3} < \infty \quad (\text{or} \quad \dim \mathcal{M}[P(z)]_{W_1 W_2}^{W_3} < \infty).$$

The natural evaluation map

$$\begin{aligned} W_1 \otimes W_2 \otimes \mathcal{M}[P(z)]_{W_1 W_2}^{W_3} &\rightarrow \overline{W}_3 \\ w_{(1)} \otimes w_{(2)} \otimes F &\mapsto F(w_{(1)} \otimes w_{(2)}) \end{aligned} \quad (12.8)$$

gives a natural map

$$\mathcal{F}[P(z)]_{W_1 W_2}^{W_3} : W_1 \otimes W_2 \rightarrow (\mathcal{M}[P(z)]_{W_1 W_2}^{W_3})^* \otimes \overline{W}_3. \quad (12.9)$$

Since $\dim \mathcal{M}[P(z)]_{W_1 W_2}^{W_3} < \infty$, $(\mathcal{M}[P(z)]_{W_1 W_2}^{W_3})^* \otimes W_3$ is a V -module (with finite-dimensional weight spaces) in the obvious way, and the map $\mathcal{F}[P(z)]_{W_1 W_2}^{W_3}$ is clearly a $P(z)$ -intertwining map, where we make the identification

$$(\mathcal{M}[P(z)]_{W_1 W_2}^{W_3})^* \otimes \overline{W}_3 = \overline{(\mathcal{M}[P(z)]_{W_1 W_2}^{W_3})^* \otimes W_3}. \quad (12.10)$$

This gives us a natural $P(z)$ -product.

The next result shows that $P(z)$ -tensor products exist for the category of modules for a rational vertex operator algebra (recall Definition I.6). It is proved by the same argument used to prove the analogous result in [HL2] for the $Q(z)$ -tensor product. As in the case of $Q(z)$ -tensor products, there is no need to assume that W_1 and W_2 are irreducible in the formulation or proof, but by Proposition 12.4, the case in which W_1 and W_2 are irreducible gives all the necessary information, and the tensor product is canonically described using only the spaces of intertwining maps among triples of *irreducible* modules.

Proposition 12.5 *Let V be rational and let W_1, W_2 be V -modules. Then*

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$$

exists, and in fact

$$W_1 \boxtimes_{P(z)} W_2 = \prod_{i=1}^k (\mathcal{M}[P(z)]_{W_1 W_2}^{M_i})^* \otimes M_i, \quad (12.11)$$

where $\{M_1, \dots, M_k\}$ is a set of representatives of the equivalence classes of irreducible V -modules, and the right-hand side of (12.11) is equipped with the V -module and $P(z)$ -product structure indicated above. That is,

$$\boxtimes_{P(z)} = \sum_{i=1}^k \mathcal{F}[P(z)]_{W_1 W_2}^{M_i}. \quad \square \quad (12.12)$$

Remark 12.6 *By combining Proposition 12.5 with Proposition 12.2, we can express $W_1 \boxtimes_{P(z)} W_2$ in terms of $\mathcal{V}_{W_1 W_2}^{M_i}$ in place of $\mathcal{M}[P(z)]_{W_1 W_2}^{M_i}$.*

The construction in Proposition 12.5 is tautological, and we view the argument as essentially an existence proof. We shall give two constructions of a $P(z)$ -tensor product, under suitable conditions, in the next section.

13 Constructions of $P(z)$ -tensor product

In this section, using the results of [HL2] and [HL3] (see also the introduction to the present paper), we give two constructions of a $P(z)$ -tensor product of two V -modules W_1 and W_2 when certain conditions are satisfied. This treatment is parallel to that of the $Q(z)$ -tensor product $W_1 \boxtimes_{Q(z)} W_2$ of W_1 and W_2 . In particular, when the vertex operator algebra is rational, we construct a $P(z)$ -tensor product of W_1 and W_2 in ways that are more useful than Proposition 12.5.

Combining Proposition I.1 (Proposition 4.7 in [HL2]), Proposition I.3 (Proposition 4.9 in [HL2]) and Proposition 12.2, we obtain an isomorphism from the space $\mathcal{M}[Q(z)]_{W_1 W_2}^{W_1 \boxtimes_{Q(z)} W_2}$ of $Q(z)$ -intertwining maps to the space $\mathcal{M}[P(z)]_{W_1 W_2}^{W_1 \boxtimes_{Q(z)} W_2}$ of $P(z)$ -intertwining maps for any pair (p_1, p_2) of integers, when $W_1 \boxtimes_{Q(z)} W_2$ exists. Thus for any pair (p_1, p_2) of integers, we have a $P(z)$ -product consisting of the module $W_1 \boxtimes_{Q(z)} W_2$ and the $P(z)$ -intertwining map which is the image of the $Q(z)$ -intertwining map $\boxtimes_{Q(z)}$ under the isomorphism corresponding to (p_1, p_2) . It is easy to show that this is a $P(z)$ -tensor product of W_1 and W_2 . Since we shall be interested in associativity and other nice properties, we shall not discuss in detail the proof that the $P(z)$ -intertwining map above gives a $P(z)$ -tensor product. Instead, we would like to construct a $P(z)$ -tensor product in a way analogous to the construction of the $Q(z)$ -tensor product in Section 5 and 6. To construct such a $P(z)$ -tensor product of two modules, we could first prove results analogous to those used in the constructions of the $Q(z)$ -tensor product, and then obtain the $P(z)$ -tensor product. But there is an easier way: We can use some of the results proved for the construction of the $Q(z^{-1})$ -tensor product (mainly the results in Section 9, 10 and 11) to derive the results that we want for the construction of the $P(z)$ -tensor product. This is what we shall do in this section. We chose to construct the $Q(z)$ -tensor product first in the present series of papers because in the case that $z = 1$ and W_1, W_2 are modules for a vertex operator algebra associated to an affine Lie algebra, it can be proved that the $Q(1)$ -tensor product of W_1 and W_2 agrees with the tensor product constructed by Kazhdan and Lusztig in [KL1]. However, from the geometric viewpoint, the simplest case of the associativity is the one for $P(z)$ -tensor products ($z \in \mathbb{C}^\times$), although in the proof of this associativity, $Q(z)$ -tensor products are also used [H]. The interested reader can imitate the constructions and calculations in

[HL2] and [HL3] to obtain the $P(z)$ -tensor product directly. We emphasize, though, that the $P(z)$ - and $Q(z)$ -tensor products are on equal footing.

For two V -modules (W_1, Y_1) and (W_2, Y_2) , we define a linear action of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

on $(W_1 \otimes W_2)^*$ (recall that ι_+ denotes the operation of expansion of a rational function of t in the direction of positive powers of t), that is, a linear map

$$\tau_{P(z)} : V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \rightarrow \text{End}(W_1 \otimes W_2)^*, \quad (13.1)$$

by

$$\begin{aligned} & \left(\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \lambda \right) (w_{(1)} \otimes w_{(2)}) = \\ & = z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\ & + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2^*(v, x_1) w_{(2)}) \end{aligned} \quad (13.2)$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. The formula (13.2) does indeed give a well-defined map of the type (13.1) (in generating-function form); this definition is motivated by (12.1), in which we first replace x_1 by x_1^{-1} and then replace v by $e^{x_1 L(1)}(-x_1^{-2})^{L(0)} v$.

Let W_3 be another V -module. The space $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ acts on W'_3 in the obvious way, where $v \otimes t^n$ ($v \in V$, $n \in \mathbb{Z}$) acts as the component v_n of $Y(v, x)$. The following result, which follows immediately from the definitions (12.1) and (13.2) and is analogous to Proposition I.9 (Proposition 5.3 in [HL2]), provides further motivation for the definition of our action on $(W_1 \otimes W_2)^*$:

Proposition 13.1 *Under the natural isomorphism*

$$\text{Hom}(W'_3, (W_1 \otimes W_2)^*) \xrightarrow{\sim} \text{Hom}(W_1 \otimes W_2, \overline{W}_3), \quad (13.3)$$

the maps in $\text{Hom}(W'_3, (W_1 \otimes W_2)^)$ intertwining the two actions of*

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

on W'_3 and $(W_1 \otimes W_2)^$ correspond exactly to the $P(z)$ -intertwining maps of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$.* \square

Remark 13.2 Combining the last result with Proposition 12.2, we see that the maps in $\text{Hom}(W'_3, (W_1 \otimes W_2)^*)$ intertwining the two actions on W'_3 and $(W_1 \otimes W_2)^*$ also correspond exactly to the intertwining operators of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$. In particular, for any intertwining operator \mathcal{Y} of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$ and any integer p , the map $F'_{\mathcal{Y}, p} : W'_3 \rightarrow (W_1 \otimes W_2)^*$ defined by

$$F'_{\mathcal{Y}, p}(w'_{(3)})(w_{(1)} \otimes w_{(2)}) = \langle w'_{(3)}, \mathcal{Y}(w_{(1)}, e^{l_p(z)} w_{(2)}) \rangle_{W_3} \quad (13.4)$$

for $w'_{(3)} \in W'_3$ (recall (12.3)) intertwines the actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ on W'_3 and $(W_1 \otimes W_2)^*$.

Write

$$Y'_{P(z)}(v, x) = \tau_{P(z)}(Y_t(v, x)) \quad (13.5)$$

(the specialization of (13.2) to $V \otimes \mathbb{C}[t, t^{-1}]$). More explicitly (as in formula (5.4) of Part I), (13.2) gives:

$$\begin{aligned} & (Y'_{P(z)}(v, x)\lambda)(w_{(1)} \otimes w_{(2)}) = \\ &= \text{Res}_{x_0} z^{-1} \delta\left(\frac{x^{-1} - x_0}{z}\right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\ &+ \text{Res}_{x_0} x_0^{-1} \delta\left(\frac{z - x^{-1}}{-x_0}\right) \lambda(w_{(1)} \otimes Y_2^*(v, x)w_{(2)}) \\ &= \text{Res}_{x_0} z^{-1} \delta\left(\frac{x^{-1} - x_0}{z}\right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\ &+ \lambda(w_{(1)} \otimes Y_2^*(v, x)w_{(2)}) \end{aligned} \quad (13.6)$$

We have the following straightforward result, as in Proposition 5.1 of Part I:

Proposition 13.3 *The action $Y'_{P(z)}$ of $V \otimes \mathbb{C}[t, t^{-1}]$ on $(W_1 \otimes W_2)^*$ has the property*

$$Y'_{P(z)}(\mathbf{1}, x) = 1, \quad (13.7)$$

where 1 on the right-hand side is the identity map of $(W_1 \otimes W_2)^*$, and the $L(-1)$ -derivative property

$$\frac{d}{dx} Y'_{P(z)}(v, x) = Y'_{P(z)}(L(-1)v, x) \quad (13.8)$$

for $v \in V$.

Proof The first part follows directly from the definition. We prove the $L(-1)$ -derivative property. From (13.6), we obtain

$$\begin{aligned}
& \left(\frac{d}{dx} Y'_{P(z)}(v, x) \lambda \right) (w_{(1)} \otimes w_{(2)}) = \\
& = \frac{d}{dx} \operatorname{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + \frac{d}{dx} \lambda(w_{(1)} \otimes Y_2^*(v, x)w_{(2)}) \\
& = \operatorname{Res}_{x_0} \frac{d}{dx} \left(z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + \operatorname{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \frac{d}{dx} \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + \lambda(w_{(1)} \otimes \frac{d}{dx} Y_2^*(v, x)w_{(2)}) \\
& = \operatorname{Res}_{x_0} \frac{d}{dx} \left(z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + \operatorname{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}L(1)(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad - 2 \operatorname{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{xL(1)}L(0)x^{-1}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + \lambda(w_{(1)} \otimes Y_2^*(L(-1)v, x)w_{(2)}). \tag{13.9}
\end{aligned}$$

The first term on the right-hand side of (13.9) is equal to

$$\begin{aligned}
& - \operatorname{Res}_{x_0} x^{-2} \frac{d}{dx^{-1}} \left(z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = \operatorname{Res}_{x_0} x^{-2} \frac{d}{dx_0} \left(z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \right) \cdot \\
& \quad \cdot \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
& = - \operatorname{Res}_{x_0} x^{-2} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot
\end{aligned}$$

$$\begin{aligned}
& \cdot \frac{d}{dx_0} \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}) \\
&= - \operatorname{Res}_{x_0} x^{-2} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \cdot \\
& \quad \cdot \lambda(Y_1(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}), \quad (13.10)
\end{aligned}$$

where we have used the “integration by parts” property of Res_{x_0} . By (13.10), (5.2.14) of [FHL] and an appropriate analogue of (5.2.12) of [FHL], the right-hand side of (13.9) is equal to

$$\begin{aligned}
& \operatorname{Res}_{x_0} z^{-1} \delta \left(\frac{x^{-1} - x_0}{z} \right) \lambda(Y_1(e^{xL(1)}(-x^{-2})^{L(0)}L(-1)v, x_0)w_{(1)} \otimes w_{(2)}) \\
& \quad + \lambda(w_{(1)} \otimes Y_2^*(L(-1)v, x)w_{(2)}) \\
&= (Y'_{P(z)}(L(-1)v, x)\lambda)(w_{(1)} \otimes w_{(2)}),
\end{aligned}$$

proving the $L(-1)$ -derivative property. \square

Write

$$Y'_{P(z)}(\omega, x) = \sum_{n \in \mathbb{Z}} L'_{P(z)}(n) x^{-n-2} \quad (13.11)$$

(recall that ω is the generator of the Virasoro algebra, for our vertex operator algebra $(V, Y, \mathbf{1}, \omega)$). We call the eigenspaces of the operator $L'_{P(z)}(0)$ the $P(z)$ -weight subspaces or $P(z)$ -homogeneous subspaces of $(W_1 \otimes W_2)^*$, and we have the corresponding notions of $P(z)$ -weight vector (or $P(z)$ -homogeneous vector) and $P(z)$ -weight.

We shall not discuss commutators $[Y'_{P(z)}(v_1, x_1), Y'_{P(z)}(v_2, x_2)]$ on $(W_1 \otimes W_2)^*$ (cf. Proposition 5.2 of Part I), but we shall instead directly discuss the “compatibility condition,” which will lead to the Jacobi identity on a suitable subspace of $(W_1 \otimes W_2)^*$.

Suppose that $G \in \operatorname{Hom}(W'_3, (W_1 \otimes W_2)^*)$ intertwines the two actions as in Proposition 13.1. Then for $w'_{(3)} \in W'_3$, $G(w'_{(3)})$ satisfies the following nontrivial and subtle condition on $\lambda \in (W_1 \otimes W_2)^*$: The formal Laurent series $Y'_{P(z)}(v, x_0)\lambda$ involves only finitely many negative powers of x_0 and

$$\begin{aligned}
& \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \lambda = \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}(v, x_1) \lambda \quad \text{for all } v \in V. \quad (13.12)
\end{aligned}$$

(Note that the two sides are not *a priori* equal for general $\lambda \in (W_1 \otimes W_2)^*$.) We call this the $P(z)$ -compatibility condition on $\lambda \in (W_1 \otimes W_2)^*$. (Note that this compatibility condition is different from the compatibility condition for $\tau_{Q(z)} \cdot$)

Let W be a subspace of $(W_1 \otimes W_2)^*$. We say that W is $P(z)$ -compatible if every element of W satisfies the $P(z)$ -compatibility condition. Also, we say that W is (\mathbb{C}) -graded (by $L'_{P(z)}(0)$) if it is \mathbb{C} -graded by its $P(z)$ -weight subspaces, and that W is a V -module (respectively, *generalized V -module*) if W is graded and is a module (respectively, generalized module) when equipped with this grading and with the action of $Y'_{P(z)}(\cdot, x)$. A sum of compatible modules or generalized modules is clearly a generalized module. The weight subspace of a subspace W with weight $n \in \mathbb{C}$ will be denoted $W_{(n)}$.

Given G as above, it is clear that $G(W'_3)$ is a V -module since G intertwines the two actions of $V \otimes \mathbb{C}[t, t^{-1}]$. We have in fact established that $G(W'_3)$ is in addition a $P(z)$ -compatible V -module since G intertwines the full actions. Moreover, if $H \in \text{Hom}(W'_3, (W_1 \otimes W_2)^*)$ intertwines the two actions of $V \otimes \mathbb{C}[t, t^{-1}]$, then H intertwines the two actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$ if and only if the V -module $H(W'_3)$ is $P(z)$ -compatible.

Define

$$W_1 \boxtimes_{P(z)} W_2 = \sum_{W \in \mathcal{W}_{P(z)}} W = \bigcup_{W \in \mathcal{W}_{P(z)}} W \subset (W_1 \otimes W_2)^*, \quad (13.13)$$

where $\mathcal{W}_{P(z)}$ is the set of all $P(z)$ -compatible modules in $(W_1 \otimes W_2)^*$. Then $W_1 \boxtimes_{P(z)} W_2$ is a $P(z)$ -compatible generalized module and coincides with the sum (or union) of the images $G(W'_3)$ of modules W'_3 under the maps G as above. Moreover, for any V -module W_3 and any map $H : W'_3 \rightarrow W_1 \boxtimes_{P(z)} W_2$ of generalized modules, $H(W'_3)$ is $P(z)$ -compatible and hence H intertwines the two actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$. Thus we have:

Proposition 13.4 *The subspace $W_1 \boxtimes_{P(z)} W_2$ of $(W_1 \otimes W_2)^*$ is a generalized module with the following property: Given any V -module W_3 , there is a natural linear isomorphism determined by (13.3) between the space of all $P(z)$ -intertwining maps of type $\begin{pmatrix} W_3 \\ W_1 \otimes W_2 \end{pmatrix}$ and the space of all maps of generalized modules from W'_3 to $W_1 \boxtimes_{P(z)} W_2$. \square*

For rational vertex operator algebras, we have the following straightforward result, proved exactly as in Proposition 5.6 of Part I:

Proposition 13.5 *Let V be a rational vertex operator algebra and W_1, W_2 two V -modules. Then the generalized module $W_1 \boxtimes_{P(z)} W_2$ is a module. \square*

Now we assume that $W_1 \boxtimes_{P(z)} W_2$ is a module (which occurs if V is rational, by the last proposition). In this case, we define a V -module $W_1 \boxtimes_{P(z)} W_2$ by

$$W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxtimes_{P(z)} W_2)' \quad (13.14)$$

and we write the corresponding action as $Y_{P(z)}$. Applying Proposition 13.4 to the special module $W_3 = W_1 \boxtimes_{P(z)} W_2$ and the identity map $W_3' \rightarrow W_1 \boxtimes_{P(z)} W_2$, we obtain using (13.3) a canonical $P(z)$ -intertwining map of type $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$, which we denote

$$\begin{aligned} \boxtimes_{P(z)} : W_1 \otimes W_2 &\rightarrow \overline{W_1 \boxtimes_{P(z)} W_2} \\ w_{(1)} \otimes w_{(2)} &\mapsto w_{(1)} \boxtimes_{P(z)} w_{(2)}. \end{aligned} \quad (13.15)$$

It is easy to verify, just as in Proposition 5.7 of Part I:

Proposition 13.6 *The $P(z)$ -product $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ is a $P(z)$ -tensor product of W_1 and W_2 . \square*

More generally, dropping the assumption that $W_1 \boxtimes_{P(z)} W_2$ is a module, we have, by imitating the proof of Proposition 5.8 of Part I:

Proposition 13.7 *The $P(z)$ -tensor product of W_1 and W_2 exists (and is given by (13.14)) if and only if $W_1 \boxtimes_{P(z)} W_2$ is a module. \square*

We observe that any element of $W_1 \boxtimes_{P(z)} W_2$ is an element λ of $(W_1 \otimes W_2)^*$ satisfying:

The $P(z)$ -compatibility condition (recall (13.12))

(a) The $P(z)$ -lower truncation condition: For all $v \in V$, the formal Laurent series $Y'_{P(z)}(v, x)\lambda$ involves only finitely many negative powers of x .

(b) The following formula holds:

$$\begin{aligned} \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \lambda = \\ = x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}(v, x_1) \lambda \quad \text{for all } v \in V. \end{aligned} \quad (13.16)$$

The $P(z)$ -local grading-restriction condition

(a) The $P(z)$ -grading condition: λ is a (finite) sum of weight vectors of $(W_1 \otimes W_2)^*$.

(b) Let W_λ be the smallest subspace of $(W_1 \otimes W_2)^*$ containing λ and stable under the component operators $\tau_{P(z)}(v \otimes t^n)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $n \in \mathbb{Z}$. Then the weight spaces $(W_\lambda)_{(n)}$, $n \in \mathbb{C}$, of the (graded) space W_λ have the properties

$$\dim (W_\lambda)_{(n)} < \infty \quad \text{for } n \in \mathbb{C}, \quad (13.17)$$

$$(W_\lambda)_{(n)} = 0 \quad \text{for } n \text{ whose real part is sufficiently small.} \quad (13.18)$$

We shall call the compatibility condition, the lower truncation condition, the local grading-restriction condition and the grading condition for $\tau_{Q(z)}$ the $Q(z)$ -compatibility condition, the $Q(z)$ -lower truncation condition, the $Q(z)$ -local grading-restriction condition and the $Q(z)$ -grading condition when it is necessary to distinguish those conditions for $\tau_{P(z)}$ and for $\tau_{Q(z)}$.

The next lemma allows us to use the results proved in the construction of the $Q(z^{-1})$ -tensor product to give another, much more useful, construction of the $P(z)$ -tensor product. Let $\psi : W_1 \otimes W_2 \rightarrow W_1 \otimes W_2$ be the linear map defined by

$$\psi(w_{(1)} \otimes w_{(2)}) = e^{-z^{-1}L(1)}w_{(1)} \otimes e^{(-2\log z + \pi i)L(0)}e^{z^{-1}L(1)}w_{(2)} \quad (13.19)$$

where $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ and let $\psi^* : (W_1 \otimes W_2)^* \rightarrow (W_1 \otimes W_2)^*$ be the adjoint of ψ . Note that ψ is a linear isomorphism and hence so is ψ^* .

Lemma 13.8 *For any $f \in (W_1 \otimes W_2)^*$, we have*

$$\begin{aligned} \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \psi^*(f) &= \\ = (zx_0)^{-1} \psi^* \left(\tau_{Q(z^{-1})} \left(zx_0 x_1 \delta \left(\frac{z^{-1} + x_0^{-1}}{(zx_0 x_1)^{-1}} \right) \right. \right. \\ \left. \left. \cdot Y_t(e^{zx_0 x_1 L(1)}(x_0 x_1)^{-2L(0)}v, x_0^{-1}) \right) f \right). \end{aligned} \quad (13.20)$$

Proof For any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, by the definitions of $\tau_{P(z)}$ and ψ and by the properties of the formal δ -functions, we have

$$\begin{aligned}
& \left(\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \psi^*(f) \right) (w_{(1)} \otimes w_{(2)}) = \\
& = z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) (\psi^*(f)) (Y_1(e^{x_1 L(1)} (-x_1^2)^{-L(0)} v, x_0) w_{(1)} \otimes w_{(2)}) \\
& \quad + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) (\psi^*(f)) (w_{(1)} \otimes Y_2^*(v, x_1) w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) f(e^{-z^{-1} L(1)} Y_1(e^{x_1 L(1)} (-x_1^2)^{-L(0)} v, x_0) w_{(1)} \otimes \\
& \quad \otimes e^{(-2 \log z + \pi i) L(0)} e^{z^{-1} L(1)} w_{(2)}) \\
& \quad + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) f(e^{-z^{-1} L(1)} w_{(1)} \otimes \\
& \quad \otimes e^{(-2 \log z + \pi i) L(0)} e^{z^{-1} L(1)} Y_2^*(v, x_1) w_{(2)}). \tag{13.21}
\end{aligned}$$

As in the proof of Lemma 5.2.3 of [FHL], together with the use of the adjoints, acting on W'_2 , of the operators $Y_2^*(v, x)$, $L(0)$ and $L(1)$ acting on W_2 (recall (3.22) in Part I and (5.2.10) in [FHL]), we have

$$e^{\zeta L(1)} Y_1(v, x) e^{-\zeta L(1)} = Y_1(e^{\zeta(1-\zeta x) L(1)} (1 - \zeta x)^{-2L(0)} v, x/(1 - \zeta x)), \tag{13.22}$$

$$e^{\zeta L(1)} Y_2^*(v, x) e^{-\zeta L(1)} = Y_2^*(v, x - \zeta), \tag{13.23}$$

$$e^{\zeta L(0)} Y_2^*(v, x) e^{-\zeta L(0)} = Y^*(e^{-\zeta L(0)} v, e^{-\zeta x}) \tag{13.24}$$

for any complex number ζ . Using (13.22)–(13.24), we see that the right-hand side of (13.21) becomes

$$\begin{aligned}
& z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) f(Y_1(e^{-z^{-1}(1+z^{-1}x_0)L(1)} (1 + z^{-1}x_0)^{-2L(0)} \cdot \\
& \quad \cdot e^{x_1 L(1)} (-x_1^2)^{-L(0)} v, x_0/(1 + z^{-1}x_0)) e^{-z^{-1} L(1)} w_{(1)} \otimes \\
& \quad \otimes e^{(-2 \log z + \pi i) L(0)} e^{z^{-1} L(1)} w_{(2)}) \\
& \quad + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) f(e^{-z^{-1} L(1)} w_{(1)} \otimes \\
& \quad \otimes Y_2^*((-z^2)^{L(0)} v, -z^2(x_1 - z^{-1})) e^{(-2 \log z + \pi i) L(0)} e^{z^{-1} L(1)} w_{(2)}). \tag{13.25}
\end{aligned}$$

Using the properties of the formal δ -functions, (13.25) is equal to

$$\begin{aligned}
& x_1 \delta \left(\frac{z + x_0}{x_1^{-1}} \right) f(Y_1(e^{-z^{-2}x_1^{-1}L(1)}(zx_1)^{2L(0)}e^{x_1L(1)} \\
& \quad \cdot (-x_1^2)^{-L(0)}v, zx_0x_1)e^{-z^{-1}L(1)}w_{(1)} \otimes e^{(-2\log z + \pi i)L(0)}e^{z^{-1}L(1)}w_{(2)}) \\
& + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) f(e^{-z^{-1}L(1)}w_{(1)} \otimes \\
& \quad \otimes Y_2^*((-z^2)^{L(0)}v, zx_0x_1)e^{(-2\log z + \pi i)L(0)}e^{z^{-1}L(1)}w_{(2)}). \tag{13.26}
\end{aligned}$$

By the formula

$$x^{L(0)}e^{x_1L(1)}x^{-L(0)} = e^{(x_1/x)L(1)} \tag{13.27}$$

(cf. (5.3.1)–(5.3.3) in [FHL]) and the definitions of Y_1^* and Y_2^* and of $\tau_{Q(z^{-1})}$, (13.26) is equal to

$$\begin{aligned}
& x_1 \delta \left(\frac{z + x_0}{x_1^{-1}} \right) f(Y_1((-z^2)^{L(0)}v, zx_0x_1)e^{-z^{-1}L(1)}w_{(1)} \otimes \\
& \quad \otimes e^{(-2\log z + \pi i)L(0)}e^{z^{-1}L(1)}w_{(2)}) \\
& + x_0^{-1} \delta \left(\frac{z - x_1^{-1}}{-x_0} \right) f(e^{-z^{-1}L(1)}w_{(1)} \otimes Y_2(e^{zx_0x_1L(1)} \cdot \\
& \quad \cdot (-zx_0x_1)^{-2})^{L(0)}(-z^2)^{L(0)}v, (zx_0x_1)^{-1})e^{(-2\log z + \pi i)L(0)}e^{z^{-1}L(1)}w_{(2)}) \\
& = z^{-1} \delta \left(\frac{x_1^{-1} - x_0}{z} \right) f(Y_1^*((-zx_0x_1)^{-2})^{L(0)}e^{-(zx_0x_1)^{-1}L(1)} \cdot \\
& \quad \cdot (-z^2)^{L(0)}v, (zx_0x_1)^{-1})e^{-z^{-1}L(1)}w_{(1)} \otimes e^{(-2\log z + \pi i)L(0)}e^{z^{-1}L(1)}w_{(2)}) \\
& - z^{-1} \delta \left(\frac{x_0 - x_1^{-1}}{-z} \right) f(e^{-z^{-1}L(1)}w_{(1)} \otimes Y_2(e^{zx_0x_1L(1)} \cdot \\
& \quad \cdot (x_0x_1)^{-2L(0)}v, (zx_0x_1)^{-1})e^{(-2\log z + \pi i)L(0)}e^{z^{-1}L(1)}w_{(2)}) \\
& = (zx_0)^{-1}x_0 \delta \left(\frac{(zx_0x_1)^{-1} - z^{-1}}{x_0^{-1}} \right) f(Y_1^*(e^{zx_0x_1L(1)}(x_0x_1)^{-2L(0)}v, \\
& \quad (zx_0x_1)^{-1})e^{-z^{-1}L(1)}w_{(1)} \otimes e^{(-2\log z + \pi i)L(0)}e^{z^{-1}L(1)}w_{(2)}) \\
& - (zx_0)^{-1}x_0 \delta \left(\frac{z^{-1} - (zx_0x_1)^{-1}}{-x_0^{-1}} \right) f(e^{-z^{-1}L(1)}w_{(1)} \otimes \\
& \quad \otimes Y_2(e^{zx_0x_1L(1)}(x_0x_1)^{-2L(0)}v, (zx_0x_1)^{-1})e^{(-2\log z + \pi i)L(0)}e^{z^{-1}L(1)}w_{(2)})
\end{aligned}$$

$$\begin{aligned}
&= (zx_0)^{-1} \psi^* \left(\tau_{Q(z^{-1})} \left(zx_0 x_1 \delta \left(\frac{z^{-1} + x_0^{-1}}{(zx_0 x_1)^{-1}} \right) \right. \right. \\
&\quad \left. \left. \cdot Y_t(e^{zx_0 x_1 L(1)}(x_0 x_1)^{-2L(0)} v, x_0^{-1}) \right) f \right) (w_{(1)} \otimes w_{(2)}), \tag{13.28}
\end{aligned}$$

proving (13.20). \square

The following result includes the analogues of Theorem 6.1 and Proposition 6.2 of [HL2], which are parts of Theorem 6.3 of [HL2] (stated as Theorem I.15 above). We state the result as one theorem since the proofs are intermeshed with one another.

Theorem 13.9 *An element $f \in (W_1 \otimes W_2)^*$ satisfies the $P(z)$ -compatibility condition if and only if $(\psi^*)^{-1}(f)$ satisfies the $Q(z^{-1})$ -compatibility condition. In this case, f satisfies the $P(z)$ -local grading-restriction condition if and only if $(\psi^*)^{-1}(f)$ satisfies the $Q(z^{-1})$ -local grading-restriction condition. If f satisfies both conditions, then $\tau_{P(z)}(v \otimes t^n)f$, $v \in V$, $n \in \mathbb{Z}$, also satisfies both conditions and we have the Jacobi identity acting on f :*

$$\begin{aligned}
&x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{P(z)}(u, x_1) Y'_{P(z)}(v, x_2) f \\
&\quad - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{P(z)}(v, x_2) Y'_{P(z)}(u, x_1) f \\
&= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y'_{P(z)}(Y(u, x_0)v, x_2) f. \tag{13.29}
\end{aligned}$$

Proof Let f be an element of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -compatibility condition. Since f satisfies the $P(z)$ -lower truncation condition,

$$Y'_{P(z)}((1 + zx_0^{-1})^{-2L(0)} v, x_0^{-1}(1 + zx_0^{-1})^{-1}) f$$

is well defined for any $v \in V$ and is a Laurent series in x_0^{-1} containing only finitely many negative powers of x_0^{-1} . From the $P(z)$ -compatibility condition and the fundamental property of the δ -function, we have

$$\begin{aligned}
&\tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t((x_0 x_1)^{2L(0)} v, x_1) \right) f \\
&= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}((x_0 x_1)^{2L(0)} v, x_1) f
\end{aligned}$$

$$= x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}((1 + zx_0^{-1})^{-2L(0)} v, x_0^{-1}(1 + zx_0^{-1})^{-1}) f. \quad (13.30)$$

Substituting $e^{-zx_0x_1L(1)}v$ for v and again using the fundamental property of the δ -function on the right-hand side, we find that

$$\begin{aligned} \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t((x_0x_1)^{2L(0)} e^{-zx_0x_1L(1)} v, x_1) \right) f = \\ = x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}((1 + zx_0^{-1})^{-2L(0)} \cdot \\ \cdot e^{-z(1+zx_0^{-1})^{-1}L(1)} v, x_0^{-1}(1 + zx_0^{-1})^{-1}) f. \end{aligned} \quad (13.31)$$

From Lemma 13.8, we have

$$\begin{aligned} \tau_{P(z)} \left(x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y_t((x_0x_1)^{2L(0)} e^{-zx_0x_1L(1)} v, x_1) \right) f = \\ = (zx_0)^{-1} \psi^* \left(\tau_{Q(z^{-1})} \left(zx_0x_1 \delta \left(\frac{z^{-1} + x_0^{-1}}{(zx_0x_1)^{-1}} \right) Y_t(v, x_0^{-1}) \right) (\psi^*)^{-1}(f) \right) \end{aligned} \quad (13.32)$$

and we obtain

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}((1 + zx_0^{-1})^{-2L(0)} \cdot \\ \cdot e^{-z(1+zx_0^{-1})^{-1}L(1)} v, x_0^{-1}(1 + zx_0^{-1})^{-1}) f \\ = (zx_0)^{-1} \psi^* \left(\tau_{Q(z^{-1})} \left(zx_0x_1 \delta \left(\frac{z^{-1} + x_0^{-1}}{(zx_0x_1)^{-1}} \right) Y_t(v, x_0^{-1}) \right) (\psi^*)^{-1}(f) \right). \end{aligned} \quad (13.33)$$

Extracting $\text{Res}_{x_1^{-1}}$ gives

$$\begin{aligned} Y'_{P(z)}((1 + zx_0^{-1})^{-2L(0)} e^{-z(1+zx_0^{-1})^{-1}L(1)} v, x_0^{-1}(1 + zx_0^{-1})^{-1}) f \\ = \text{Res}_{x_1^{-1}}(zx_0)^{-1} \cdot \\ \cdot \psi^* \left(\tau_{Q(z^{-1})} \left(zx_0x_1 \delta \left(\frac{z^{-1} + x_0^{-1}}{(zx_0x_1)^{-1}} \right) Y_t(v, x_0^{-1}) \right) (\psi^*)^{-1}(f) \right) \end{aligned}$$

$$\begin{aligned}
&= \text{Res}_y \psi^* \left(\tau_{Q(z^{-1})} \left(y^{-1} \delta \left(\frac{z^{-1} + x_0^{-1}}{y} \right) Y_t(v, x_0^{-1}) \right) (\psi^*)^{-1}(f) \right) \\
&= \psi^* (Y'_{Q(z^{-1})}(v, x_0^{-1}) (\psi^*)^{-1}(f)). \tag{13.34}
\end{aligned}$$

Since the left-hand side of (13.34) is a well-defined formal Laurent series in x_0^{-1} containing only finitely many negative powers of x_0^{-1} , the right-hand side of (13.34) is also such a Laurent series. Thus $(\psi^*)^{-1}(f)$ satisfies the $Q(z^{-1})$ -lower truncation condition. Using (13.33) and (13.34), we have

$$\begin{aligned}
&(zx_0)^{-1} \psi^* \left(\tau_{Q(z^{-1})} \left(zx_0 x_1 \delta \left(\frac{z^{-1} + x_0^{-1}}{(zx_0 x_1)^{-1}} \right) Y_t(v, x_0^{-1}) \right) (\psi^*)^{-1}(f) \right) \\
&= x_1 \delta \left(\frac{x_0 + z}{x_1^{-1}} \right) \psi^* (Y'_{Q(z^{-1})}(v, x_0^{-1}) (\psi^*)^{-1}(f)), \tag{13.35}
\end{aligned}$$

or equivalently, replacing x_0 by x_0^{-1} and then replacing x_1 by $z^{-1}x_0x_1^{-1}$ and applying $(\psi^*)^{-1}$,

$$\begin{aligned}
&\tau_{Q(z^{-1})} \left(x_1^{-1} \delta \left(\frac{z^{-1} + x_0}{x_1} \right) Y_t(v, x_0) \right) (\psi^*)^{-1}(f) = \\
&= x_1^{-1} \delta \left(\frac{z^{-1} + x_0}{x_1} \right) Y'_{Q(z^{-1})}(v, x_0) (\psi^*)^{-1}(f). \tag{13.36}
\end{aligned}$$

Thus $(\psi^*)^{-1}(f)$ satisfies the $Q(z^{-1})$ -compatibility condition.

Conversely, if $(\psi^*)^{-1}(f)$ satisfies the $Q(z^{-1})$ -compatibility condition, an analogue of the proof above shows that f satisfies the $P(z)$ -compatibility condition: The $Q(z^{-1})$ -compatibility condition (13.36) and Lemma 13.8 give analogues of (13.33) and (13.34), and these analogues imply that f satisfies the $P(z)$ -compatibility condition. This finishes the proof of the first part of the theorem.

Now assume that f satisfies the $P(z)$ -compatibility condition or equivalently, that $(\psi^*)^{-1}(f)$ satisfies the $Q(z^{-1})$ -compatibility condition. By Proposition 5.1 and Theorem 6.1 of [HL2], which are parts of Theorem I.15 above (Theorem 6.3 of [HL2]), we have the $L(-1)$ -derivative property

$$\frac{d}{dx} Y'_{Q(z^{-1})}(v, x) (\psi^*)^{-1}(f) = Y'_{Q(z^{-1})}(L(-1)v, x) (\psi^*)^{-1}(f) \tag{13.37}$$

(and this in fact holds for any $g \in (W_1 \otimes W_2)^*$ in place of $(\psi^*)^{-1}(f)$) and the Jacobi identity

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{Q(z^{-1})}(u, x_1) Y'_{Q(z^{-1})}(v, x_2) (\psi^*)^{-1}(f) \\
& - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{Q(z^{-1})}(v, x_2) Y'_{Q(z^{-1})}(u, x_1) (\psi^*)^{-1}(f) \\
& = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y'_{Q(z^{-1})}(Y(u, x_0)v, x_2) (\psi^*)^{-1}(f). \tag{13.38}
\end{aligned}$$

for $u, v \in V$. Since the Virasoro element $\omega = L(-2)\mathbf{1}$ is quasi-primary, that is, $L(1)\omega = 0$, and its weight is 2, (13.34) with $v = \omega$ gives

$$\begin{aligned}
& \sum_{n \in \mathbb{Z}} (L'_{P(z)}(n)f) x_0^{n+2} (1 + zx_0^{-1})^{n-2} = \\
& = Y'_{P(z)}((1 + zx_0^{-1})^{-2L(0)} e^{-z(1+zx_0^{-1})^{-1}L(1)} \omega, x_0^{-1}(1 + zx_0^{-1})^{-1}) f \\
& = \psi^*(Y'_{Q(z^{-1})}(\omega, x_0^{-1})(\psi^*)^{-1}(f)) \\
& = \sum_{n \in \mathbb{Z}} \psi^*(L'_{Q(z^{-1})}(n)(\psi^*)^{-1}(f)) x_0^{n+2}. \tag{13.39}
\end{aligned}$$

where $L'_{P(z)}(n)$, $n \in \mathbb{Z}$, are defined in (13.11) and $L'_{Q(z^{-1})}(n)$, $n \in \mathbb{Z}$, are defined analogously as the coefficients of the vertex operator $Y'_{Q(z^{-1})}(\omega, x)$. Taking the coefficient of x_0^3 on both sides of (13.39) and noting that for $n \in \mathbb{Z}$, the coefficient of x_0^3 in $x_0^{n+2}(1 + zx_0^{-1})^{n-2}$ is $\delta_{n,1}$, we obtain

$$L'_{P(z)}(1)f = \psi^*(L'_{Q(z^{-1})}(1)(\psi^*)^{-1}(f))$$

or equivalently,

$$(\psi^*)^{-1}(L'_{P(z)}(1)f) = L'_{Q(z^{-1})}(1)(\psi^*)^{-1}(f). \tag{13.40}$$

Similarly, taking the coefficient of x_0^2 on both sides of (13.39) and using (13.40), we have

$$(\psi^*)^{-1}(L'_{P(z)}(0)f) = (L'_{Q(z^{-1})}(0) + zL'_{Q(z^{-1})}(1))(\psi^*)^{-1}(f). \tag{13.41}$$

From the Jacobi identity (13.38) together with (13.37), we obtain the usual commutator formulas for

$$[L'_{Q(z^{-1})}(m), L'_{Q(z^{-1})}(n)]$$

and for

$$[L'_{Q(z^{-1})}(m), Y'_{Q(z^{-1})}(v, x)]$$

$(m, n \in \mathbb{Z}, v \in V)$, acting on $(\psi^*)^{-1}(f)$. In particular, we have

$$[L'_{Q(z^{-1})}(0), L'_{Q(z^{-1})}(1)](\psi^*)^{-1}(f) = -L'_{Q(z^{-1})}(1)(\psi^*)^{-1}(f) \quad (13.42)$$

and

$$\begin{aligned} & [L'_{Q(z^{-1})}(1), Y'_{Q(z^{-1})}(v, x)](\psi^*)^{-1}(f) = \\ & = Y'_{Q(z^{-1})}((L(1) + 2xL(0) + x^2L(-1))v, x)(\psi^*)^{-1}(f). \end{aligned} \quad (13.43)$$

By Proposition 6.2 of [HL2] (part of Theorem I.15 above), the subspace of $(W_1 \otimes W_2)^*$ consisting of the elements satisfying the $Q(z^{-1})$ -compatibility condition is stable under the operators $\tau_{Q(z^{-1})}(v \otimes t^n)$ for $v \in V$ and $n \in \mathbb{Z}$, and in particular, under the operators $L'_{Q(z^{-1})}(n)$. Thus by (13.40), (13.41) and what has been proved above, the subspace of $(W_1 \otimes W_2)^*$ consisting of the elements satisfying the $P(z)$ -compatibility condition is stable under the operator $L'_{P(z)}(1)$ and $L'_{P(z)}(0)$. Also, by (13.40)–(13.42), we have

$$[L'_{P(z)}(0), L'_{P(z)}(1)]f = -L'_{P(z)}(1)f. \quad (13.44)$$

We also see that $\tau_{P(z)}(v \otimes t^n)$ preserves the space of elements of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -compatibility condition. Indeed, replacing x_0^{-1} in (13.34) by $x(1 - zx)^{-1}$, we have

$$\begin{aligned} & Y'_{P(z)}((1 - zx)^{2L(0)}e^{-z(1-zx)L(1)}v, x)f = \\ & = \psi^*(Y'_{Q(z^{-1})}(v, x(1 - zx)^{-1})(\psi^*)^{-1}(f)), \end{aligned} \quad (13.45)$$

or equivalently,

$$\begin{aligned} & Y'_{P(z)}(v, x)f = \\ & = \psi^*(Y'_{Q(z^{-1})}(e^{z(1-zx)L(1)}(1 - zx)^{-2L(0)}v, x(1 - zx)^{-1})(\psi^*)^{-1}(f)), \end{aligned} \quad (13.46)$$

and we now invoke the already-established equivalence between the compatibility conditions for f and $(\psi^*)^{-1}(f)$.

Now suppose that f satisfies the $P(z)$ -compatibility condition and that $L'_{P(z)}(1)$ acts nilpotently on f (that is, $(L'_{P(z)}(1))^n f = 0$ for large n). Then

of course $L'_{P(z)}(1)$ also acts nilpotently on such elements as $e^{-zL'_{P(z)}(1)}f$. By (13.40) and the comments above, $e^{-zL'_{Q(z-1)}(1)}(\psi^*)^{-1}(f)$ is also well defined (in the analogous sense). From (13.37) and (13.43), we obtain, as in the proof of (5.2.38) of [FHL] (which is valid here since we are using the two formal variables x_0 and x and since (13.43) remains valid with $(\psi^*)^{-1}(f)$ replaced by $L'_{Q(z-1)}(1)(\psi^*)^{-1}(f)$),

$$\begin{aligned} e^{x_0 L'_{Q(z-1)}(1)} Y'_{Q(z-1)}(v, x) e^{-x_0 L'_{Q(z-1)}(1)} (\psi^*)^{-1}(f) &= \\ &= Y'_{Q(z-1)}(e^{x_0(1-x_0x)L(1)}(1-x_0x)^{-2L(0)}v, x(1-x_0x)^{-1})(\psi^*)^{-1}(f). \end{aligned} \quad (13.47)$$

The coefficient of each monomial in x on the right-hand side of (13.47) is a (terminating) polynomial in x_0 , and so the same is true of the left-hand side. Thus we may substitute z for x_0 and we obtain

$$\begin{aligned} e^{z L'_{Q(z-1)}(1)} Y'_{Q(z-1)}(v, x) e^{-z L'_{Q(z-1)}(1)} (\psi^*)^{-1}(f) &= \\ &= Y'_{Q(z-1)}(e^{z(1-zx)L(1)}(1-zx)^{-2L(0)}v, x(1-zx)^{-1})(\psi^*)^{-1}(f). \end{aligned} \quad (13.48)$$

Combining (13.46) and (13.48), we have the following formula, which relates $Y'_{P(z)}(v, x)$ and $Y'_{Q(z-1)}(v, x)$ by conjugation, when acting on f :

$$Y'_{P(z)}(v, x)f = \psi^*(e^{zL'_{Q(z-1)}(1)}Y'_{Q(z-1)}(v, x)e^{-zL'_{Q(z-1)}(1)}(\psi^*)^{-1}(f)). \quad (13.49)$$

(Recall that we have just seen that $L'_{Q(z-1)}(1)$ acts nilpotently on $\tau_{Q(z-1)}(v \otimes t^n)e^{-zL'_{Q(z-1)}(1)}(\psi^*)^{-1}(f)$ for any $n \in \mathbb{Z}$.)

Now we assume in addition that f satisfies the $P(z)$ -grading condition. Taking $v = \omega$ in (13.49), we get

$$L'_{P(z)}(0)f = \psi^*(e^{zL'_{Q(z-1)}(1)}L'_{Q(z-1)}(0)e^{-zL'_{Q(z-1)}(1)}(\psi^*)^{-1}(f)) \quad (13.50)$$

or equivalently, applying $e^{-zL'_{P(z)}(1)}$ and $(\psi^*)^{-1}$ and using what we have established,

$$(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}L'_{P(z)}(0)f) = L'_{Q(z-1)}(0)(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}f). \quad (13.51)$$

If f is an eigenvector of $L'_{P(z)}(0)$, then $(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}f)$ is an eigenvector (with the same eigenvalue) of $L'_{Q(z^{-1})}(0)$ by (13.51). Since by assumption $L'_{P(z)}(1)$ acts nilpotently on f , $e^{zL'_{P(z)}(1)}f$ is a finite sum of eigenvectors of $L'_{P(z)}(0)$ (recall (13.44)), so that

$$(\psi^*)^{-1}(f) = (\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}(e^{zL'_{P(z)}(1)}f))$$

is also a finite sum of eigenvectors of $L'_{Q(z^{-1})}(0)$. In general, since f is a finite sum of eigenvectors of $L'_{P(z)}(0)$, on each of which $L'_{P(z)}(1)$ acts nilpotently, $(\psi^*)^{-1}(f)$ is still a finite sum of eigenvectors of $L'_{Q(z^{-1})}(0)$. That is, $(\psi^*)^{-1}(f)$ satisfies the $Q(z^{-1})$ -grading condition for $L'_{Q(z^{-1})}(0)$.

Continuing to assume that f satisfies the $P(z)$ -compatibility condition and that $L'_{P(z)}(1)$ acts nilpotently on f , we note that $e^{-zL'_{P(z)}(1)}f$ satisfies the same conditions. By the discussion above, $(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}f)$ satisfies the $Q(z^{-1})$ -compatibility condition. By (13.40) and (13.49) and the fact that the component operators of $Y'_{Q(z^{-1})}(v, x)$ preserve the space of elements of $(W_1 \otimes W_2)^*$ satisfying the $Q(z^{-1})$ -compatibility condition, we have, for any $v \in V$,

$$\begin{aligned} Y'_{P(z)}(v, x)f &= \\ &= \psi^*(e^{zL'_{Q(z^{-1})}(1)}Y'_{Q(z^{-1})}(v, x)(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}f)) \\ &= e^{zL'_{P(z)}(1)}\psi^*(Y'_{Q(z^{-1})}(v, x)(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}f)) \end{aligned} \quad (13.52)$$

(note that the action of $e^{zL'_{P(z)}(1)}$ is defined). Thus we have the equivalent formula

$$(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}Y'_{P(z)}(v, x)f) = Y'_{Q(z^{-1})}(v, x)(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}f), \quad (13.53)$$

and we know that the coefficient of each power of x in $Y'_{P(z)}(v, x)f$ satisfies the $P(z)$ -compatibility condition and the $L'_{P(z)}(1)$ -nilpotency condition. Thus by induction, we have

$$\begin{aligned} (\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}Y'_{P(z)}(v_1, x_1) \cdots Y'_{P(z)}(v_n, x_n)f) &= \\ &= Y'_{Q(z^{-1})}(v_1, x_1) \cdots Y'_{Q(z^{-1})}(v_n, x_n)(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}f), \end{aligned} \quad (13.54)$$

and in particular,

$$\begin{aligned}
(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}L'_{P(z)}(0)Y'_{P(z)}(v_1, x_1) \cdots Y'_{P(z)}(v_n, x_n)f) = \\
= L'_{Q(z^{-1})}(0)Y'_{Q(z^{-1})}(v_1, x_1) \cdots Y'_{Q(z^{-1})}(v_n, x_n)(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}f)
\end{aligned} \tag{13.55}$$

for any $n \geq 0$ and $v_1, \dots, v_n \in V$.

Now we assume that f satisfies both the $P(z)$ -compatibility and $P(z)$ -local grading-restriction conditions. By (13.44), $L'_{P(z)}(1)$ acts on f nilpotently. From the discussion above, we see that $(\psi^*)^{-1}(f)$ satisfies the $Q(z^{-1})$ -grading condition and the formulas (13.54) and (13.55) hold. Since f satisfies part (b) of the $P(z)$ -local grading-restriction condition, we see from (13.54)–(13.55) that the element

$$(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}f) \in (W_1 \otimes W_2)^*,$$

which satisfies the $Q(z^{-1})$ -grading condition, satisfies part (b) of the $Q(z^{-1})$ -local grading-restriction condition. Thus

$$e^{zL'_{Q(z^{-1})}(1)}(\psi^*)^{-1}(e^{-zL'_{P(z)}(1)}f) = (\psi^*)^{-1}(f)$$

also satisfies the $Q(z^{-1})$ -local-grading restriction condition, proving the “only if” part of the second assertion of the Theorem.

Let f satisfy both the $P(z)$ -compatibility condition and the $P(z)$ -local grading-restriction condition. We already know that the components of $Y'_{P(z)}(v, x)f$ satisfy the $P(z)$ -compatibility condition for any $v \in V$. Since $(\psi^*)^{-1}f$ satisfies both the $Q(z^{-1})$ -compatibility and $Q(z^{-1})$ -local grading-restriction conditions, the components of

$$Y'_{Q(z^{-1})}(v, x)e^{-zL'_{Q(z^{-1})}(1)}(\psi^*)^{-1}(f)$$

still satisfy these conditions. Thus by (13.54) and (13.55), with $e^{zL'_{P(z)}(1)} \circ \psi^*$ applied to both sides, we see that the components of $Y'_{P(z)}(v, x)f$ satisfy the $P(z)$ -local grading restriction-condition.

To prove the Jacobi identity (13.29), note that $(\psi^*)^{-1}(f)$ satisfies both the $Q(z^{-1})$ -compatibility and $Q(z^{-1})$ -local grading-restriction conditions, and

thus the Jacobi identity (13.38) for $Y'_{Q(z^{-1})}$ holds. From this Jacobi identity, we obtain

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) \psi^* (e^{zL'_{Q(z^{-1})}(1)} Y'_{Q(z^{-1})}(u, x_1) Y'_{Q(z^{-1})}(v, x_2) (\psi^*)^{-1}(f)) \\
& \quad - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \psi^* (e^{zL'_{Q(z^{-1})}(1)} Y'_{Q(z^{-1})}(v, x_2) Y'_{Q(z^{-1})}(u, x_1) (\psi^*)^{-1}(f)) \\
& = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \psi^* (e^{zL'_{Q(z^{-1})}(1)} Y'_{Q(z^{-1})}(Y(u, x_0)v, x_2) (\psi^*)^{-1}(f)).
\end{aligned} \tag{13.56}$$

Using (13.49) and (13.40) and the stability established in the last paragraph, we can write (13.56) as

$$\begin{aligned}
& x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y'_{P(z)}(u, x_1) Y'_{P(z)}(v, x_2) e^{zL'_{P(z)}(1)} f \\
& \quad - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y'_{P(z)}(v, x_2) Y'_{P(z)}(u, x_1) e^{zL'_{P(z)}(1)} f \\
& = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y'_{P(z)}(Y(u, x_0)v, x_2) e^{zL'_{P(z)}(1)} f.
\end{aligned} \tag{13.57}$$

Thus we have proved that the Jacobi identity holds when acting on $e^{zL'_{P(z)}(1)} f$ for any f as above. Replacing f by $e^{-zL'_{P(z)}(1)} f$, which still satisfies the conditions, we see that the Jacobi identity (13.29) holds.

It remains only to prove the “if” part of the second assertion of the Theorem. Note that in the proof above, (13.54) and (13.55) were the main equalities that we had to establish for the “only if” part; the conclusion followed immediately. From the proof of (13.54) and (13.55) above, we see that conversely, if $(\psi^*)^{-1}(f)$ satisfies both the $Q(z^{-1})$ -compatibility and $L'_{Q(z^{-1})}(1)$ -nilpotency conditions, we can prove analogously, using the Jacobi identity (13.29), that (13.54) and (13.55) with $(\psi^*)^{-1} \circ e^{-zL'_{P(z)}(1)}$ replaced by $e^{-zL'_{Q(z^{-1})}(1)} \circ (\psi^*)^{-1}$ hold. These two equalities and the $Q(z^{-1})$ -local grading-restriction condition for $(\psi^*)^{-1}(f)$ imply that f satisfies the $P(z)$ -local grading-restriction condition. \square

Proposition 13.3 and Theorem 13.9 give us another construction of the $P(z)$ -tensor product—the analogue of Theorem 6.3 of [HL2] (Theorem I.15 above):

Theorem 13.10 *The vector space consisting of all elements of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$ -compatibility condition and the $P(z)$ -local grading-restriction condition equipped with $Y'_{P(z)}$ is a generalized module and is equal to the generalized module $W_1 \boxtimes_{P(z)} W_2$. \square*

Finally we have the analogues of the last two results in [HL2] (recalled in the introduction above). The following result follows immediately from Proposition 13.7, the theorem above and the definition of $W_1 \boxtimes_{P(z)} W_2$:

Corollary 13.11 *The $P(z)$ -tensor product of W_1 and W_2 exists if and only if the subspace of $(W_1 \otimes W_2)^*$ consisting of the elements satisfying the compatibility condition and the local grading-restriction condition, equipped with $Y'_{P(z)}$, is a module. In this case, this module coincides with the module $W_1 \boxtimes_{P(z)} W_2$, and the contragredient module of this module, equipped with the $P(z)$ -intertwining map $\boxtimes_{P(z)}$, is a $P(z)$ -tensor product of W_1 and W_2 , equal to the structure $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ constructed above. \square*

From this result and Propositions 13.5 and 13.6, we have:

Corollary 13.12 *Let V be a rational vertex operator algebra and W_1, W_2 two V -modules. Then the $P(z)$ -tensor product $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ may be constructed as described in Corollary 13.11. \square*

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